Math 210C Lecture 21 Notes

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1 Group Cohomology, Simple and Semisimple Modules, and Schur's Lemma

1.1 More on group cohomology

If G is a group, we have the bar resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module:

$$\mathbb{Z}[G^{i+2}] \xrightarrow{d_{i+1}} \mathbb{Z}[G^{i+1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}$$

We get $H^i(G, A) \cong \operatorname{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, and

$$\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \xrightarrow{D^0} \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^2], A) \longrightarrow \cdots$$

We get cochains $C^i(G < A) = \{f : G^i \to A\}$, where $C^0(G, A) = A$. We have the differentials $d^i : C^i(G, A) \to C^{i+1}(G, A)$, where

$$d^0 a(g) = ga - a,$$

$$d^1 f(g, h) = gf(h) = f(gh) + f(g),$$

$$d^2 f(g, h, k) = gf(h, k) - f(gh, k) + f(g, hk) - f(g, h),$$

and so on.

We have the isomorphism $C^i(G, A) \cong \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{i+1}], A)$ by $f \mapsto \varphi$, where $\varphi(g_0, \dots, g_i) = g_0 f(g_0^{-1} g_1, g_1^{-1} g_2, \dots, g_{i-1}^{-1} g_i)$ (and extended linearly). For the inverse, given φ, define $f(g_1, \dots, g_i) = \varphi(1, g_1, g_1 g_2, \dots, g_1, \dots, g_i)$.

We have the i-cocycles $Z^i(G, A) = \ker(d^i)$ and the i-coboundaries $B^i(G, A) = \operatorname{im}(d^{i-1})$. Then we have

$$H^i(G, A) \cong \frac{Z^i(G, A)}{B^i(G, A)}.$$

We can see

$$H^0(G, A) = Z^0(G, A) = A^G,$$

$$Z^1(G,A) = \{f : G \to A \mid f(gh) = g(g) + gf(h)\ \forall g,h\},$$
$$B^1(G,A) = \{g \mapsto ga - a \mid a \in A\}.$$

If A is a trivial G-module, then $H^i(G,A) = Z^1(G,A) = \mathrm{Hom}(G,A)$. Then

$$H^2(G,A) \cong \{\text{group extensions } 1 \to A \to \mathcal{E} \to G \to 1\}/\sim.$$

If $f \in Z^2(G,A)$ with $f(g,1) = f(1,g) = 0$ for all $g \in G$, then the "factor set" mapsto $\mathcal{E}_f$, where $(a,g)(b,h) = (a + gb + f(g,h), gh)$.

If L/K is finite Galois and $G = \mathrm{Gal}(L/K)$, then $G \circlearrowleft :^\times$, so there is a $\mathbb{Z}[G]$-modules sturcture on $L^\times$. Then $H^0(G, L^\times = K^\times$. Hilbert's theorem 90 can be extended to the statement that $H^1(G, L^\times) = 0$. If G is cyclic, we get the classical Hilbert's theorem 90.

We get Kummer theory: if $\mathrm{char}(K) \nmid n$ and $\mu_n \subseteq L$, then we get an exact sequence

$$1 \longrightarrow \mu_n \longrightarrow L^\times \xrightarrow{n} L^{\times n} \longrightarrow 1$$

We get by Hilbert's theorem 90:

$$1 \longrightarrow \mu_n(K) \longrightarrow K^\times \xrightarrow{n} K^{\times n} \longrightarrow H^1(G,\mu_n) \longrightarrow 0$$

So we get

$$\mathrm{Hom}(G,\mu_n) \cong H^1(G,\mu_n) \cong K^\times \cap L^{\times n}/K^{\times n} = \Delta/K^{\times n}$$

which gives us Kummer duality. What is $H^2(G, L^\times)$?

1.2 Simple modules, semisimple modules, and Schur's lemma

Let R be a nonzero ring. Recall that R is simple if it has no nonzero, proper 2-sided ideals.

Example 1.1. Let D be a division ring. Then $M_n(D)$ is simple.

Definition 1.1. A left R-module is

1. **simple** (or **irreducible**) if it has no nonzero, proper R-submodules.
2. **indecomposable** if it is not the direct sum of two R-submodules.
3. **semisimple** (or **completely decomposable**) if it is a direct sum of simple submodules.

Remark 1.1. Being simple is the same as being the same as being semisimple and indecomposable.

Remark 1.2. Semisimple is the same as being a sum of simple submodules.

Remark 1.3. An R-module is semisimple iff every submodule is a summand of it.

Example 1.2. Let D be a division ring. Then D is a simple D-module, and $M_n(D)$ is a semisimple $M_n(D)$-module : $M_n(D) \cong (D^n)^n$, where D^n, the space of column vectors, is simple as an $M_n(D)$-module.

Example 1.3. Vector spaces are semisimple.

Example 1.4. The simple $\mathbb{Z}$-modules are $\mathbb{Z}/p\mathbb{Z}$ for p prime.

Example 1.5. $\mathbb{Z}$ is indecomposable but not semisimple.

Example 1.6. Let $R = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \subseteq M_2(F)$. Then $R \circlearrowright F^2$ and has a submodule $\begin{bmatrix} * \\ 0 \end{bmatrix}$. So it is not simple. On the other hand, if $\begin{bmatrix} a \\ b \end{bmatrix} \in F^2$ with $b \neq 0$, it generates F^2. So F^2 is indecomposable.

Lemma 1.1 (Schur). *Let R be a ring, and let M, N be simple R-modules. Then any R-module homomorphism $f : M \to N$ is either 0 or an isomorphism.*

Proof. Suppose $f \neq 0$. Then $\ker(f) \subsetneq M$, so $\ker(f) = 0$. Now $\operatorname{im}(f) \subseteq N$ and is not 0, so $\operatorname{im}(f) = N$. So f is an isomorphism. □

We can define composition series for modules as with groups. An analogue of the Jordan-Hölder theorem still holds.

Corollary 1.1. *Let M be a simple R-module. Then $\operatorname{End}_R(M)$ is a division ring.*

Proof. If $0 \neq f \in \operatorname{End}_R(M)$, then f is an isomorphism. So f has an inverse. □

Lemma 1.2. *Let M be a simple R-module, let $N \geq 1$, and let $D = \operatorname{End}_R(M)$. Then $\operatorname{End}_R(M^n) \cong M_n(D)$*

Proof. For $C \in M_n(D)$, write $C = [\phi_{i,j} : M \to M]$. Let $\Phi : M_n(D) \to \operatorname{End}_R(M)$ by $\Phi(C)(m_1, \dots, m_n) = (\sum_j \phi_{1,j}(m_j), \dots, \sum_j \phi_{n,j}(m_j))$. This is an isomorphism. □

Lemma 1.3. *Suppose $M \cong N_1^{n_1} \oplus \cdots \oplus N_k^{n_k}$, where N_i are distinct simple modules and $n_i \geq 1$. Then $\operatorname{End}_R(M) \cong \prod_{i=1}^k M_{n_i}(D_i)$ and $D_i = \operatorname{End}_R(N_i)$.*

Definition 1.2. A ring $R \neq 0$ is **semisimple** if it is semisimple as a left R-module.

Lemma 1.4. *The following are equivalent:*

1. *R is a semisimple ring.*
2. *R is a direct sum of its minimal left ideals.*
3. *R is a finite direct sum of minimal left ideals.*

Proof. Suppose $R = \sum_{i \in I} J_i$, where the J_i are minimal left ideals. We want to show that I is finite. We get $1 = j_{i_1} + \cdots + j_{i_n}$, where $i_k \in I$ and $j_{i_k} \in J_{i_k}$. Then $J_{i_1} + \cdots + J_{i_n} = R$. □